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AXISYMMETRIC VIBRATIONS OF SOLID CIRCULAR AND ANNULAR MEMBRANES WITH CONTINUOUSLY VARYING DENSITY

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The present paper reports the results from a series of numerical experiments dealing with the determination of the lower natural frequencies for the transverse vibration of annular membranes (including the special case of a solid circular membrane) when the mass per unit area varies linearly, quadratically, and cubically with the radial co-ordinate. The frequency coefficients have been determined using: (1) the differential quadrature method; (2) the finite element technique; (3) an optimized and/or improved Rayleigh quotient method; and (4) a lower bound based on the Stodola–Vianello method. Excellent agreement is achieved among the different methodologies.

1. INTRODUCTION

As stated in reference [1] "the membrane is a rather popular element in nature and in technology: from the tympanic membrane to membranic elements used in microphones and loudspeakers, pumps, compressors, etc." As rightly pointed out by Mazumdar in a very complete survey [2] "membranes have long and widely been used as transducers, devices that allow energy to change from one form to another."

When the density (mass per unit area) is constant, one has the classical Helmholtz equation once the temporal variable has been removed by separation, and the solution is well known when the boundary of the domain is such that the problem is solvable by separation of variables. The situation where the density of the membrane is not constant constitutes a more realistic situation, either as a consequence of the manufacturing process or as required by design limitations. Variations of the membrane mass per unit area (density) could be due to variations in the thickness of the membrane or variations in the material density itself, both caused by the manufacturing process. It should be noted that another possible cause of inhomogeneity in the vibrating membrane problem is non-uniform tension [3].

The analysis of vibrating membranes with varying density has been tackled by several investigators, including [1–5]. In an excellent paper, Spence and Horgan [4] found upper and lower bounds for the natural frequencies of a circular membrane with stepped radial

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density, and they showed that eigenvalue estimation techniques based on an integral equation approach are more effective than classical variational techniques.

In this paper, in addition to accurate one-term approximations using optimized and/or improved Rayleigh's quotient, the first two natural frequency coefficients corresponding to axisymmetric modes of vibration are determined using the differential quadrature (DQ) method [6] and a very convenient and simple finite element (FE) code [7].

2. GOVERNING EQUATIONS

The present paper deals with the system shown in Figure 1 for the two situations: (1) a solid circular membrane of radius R; (2) annular membrane of outer radius R and inner radius R_0.

It is assumed that the membrane density varies according to

$$\rho(\bar{r}) = \rho_0[1 + \alpha(\bar{r}/R)^n], \tag{1}$$

where $\alpha > 0$ and n has been taken equal to 1, 2 and 3 in the present study*.

Introducing the dimensionless variable

$$r = \bar{r}/R \tag{2}$$

in equation (1), one obtains

$$\rho(r) = \rho_0 f(r), \tag{3}$$

where

$$f(r) = 1 + \alpha r^n. \tag{4}$$

In the case of axisymmetric modes of vibration of an annular or a solid circular membrane of outer radius R and inner radius R_0, the governing differential equation for the displacement $W(r)$ is

$$rW'' + W' + \Omega^2 f(r) rW = 0, \qquad 0 \leqslant r_0 < r \leqslant 1, \tag{5}$$

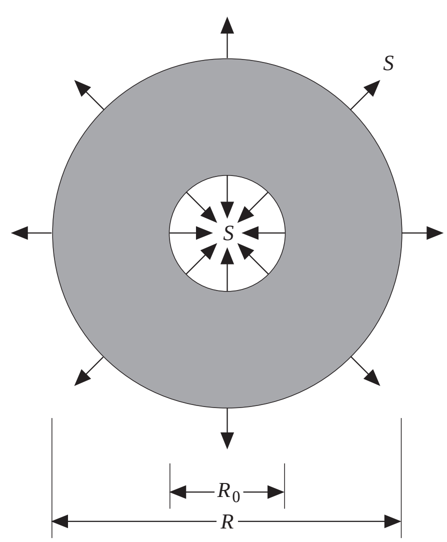

Figure 1. Vibrating annular membrane of outer radius R and inner radius R_0.

* The case where $n = 1$ has also been dealt with in reference [8].

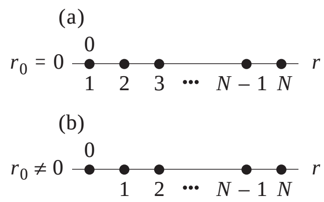

Figure 2. Partition of the intervals for (a): [0, 1] and (b): [r_0, 1].

where ( )′ is the derivative with respect to r, $r_0 = R_0/R$ and $\Omega = \omega R\sqrt{\rho_0/S}$. The boundary condition for the annular case is

$$W(r_0) = W(1) = 0, \tag{6}$$

while for the solid circular membrane the boundary condition is simply

$$W(1) = 0 \tag{7}$$

3. ONE-TERM APPROXIMATION

Considering a one-term approximation of the form $W = a\phi(r)$, one can easily show that the corresponding Rayleigh quotient governing this problem is

$$\mathscr{R} = \int_{r_0}^{1} r\phi'^2 \,\mathrm{d}r \bigg/ \int_{r_0}^{1} rf(r)\phi^2 \,\mathrm{d}r. \tag{8}$$

It is well known that functions ϕ which satisfy the geometric boundary conditions will yield a value for $\mathscr{R}$ that is an upper bound of Ω^2.

A mean for improving the accuracy of the one term Rayleigh quotient approximation using an iterative method, attributed to Stodola and Vianello, is found in reference [9]. To execute the method one solves equation (5) for the highest derivative term, regarding the right side terms as "input" and the left-hand side as "output". One can thus set up an iterative scheme of the form

$$(r\phi'_{i+1})' = -\Omega^2 f(r) r\phi_i, \qquad i = 0, 1, \ldots, \tag{9}$$

where ϕ_0 is the simplest admissible function, given by either

$$\phi_0 = 1 - r, \tag{10}$$

for the solid circular membrane, or

$$\phi_0 = (r - r_0)(1 - r) \tag{11}$$

for the annular case, both of which satisfy the geometric boundary conditions. Substituting the appropriate one of these functions into the right side of equation (9) and integrating while making use of the boundary conditions, one obtains a comparison function ϕ_1, which can be normalized as desired. These comparison functions are more complicated than ϕ_0 by far. However, the integrals required for calculation of Rayleigh's quotient can all be found in closed form with symbolic manipulation software. For the solid circular membrane case, one could start with $\phi_0 = 1 - r^2$ which also satisfies the condition $\phi'_0(0) = 0$, but it is not necessary to do so since $\phi'_1(0) = 0$ regardless of which function (i.e., $1 - r$ or $1 - r^2$) is used to start the procedure.

TABLE 1

Fundamental frequency coefficient Ω_1, *case* $\rho = \rho_0(1 + \alpha r)$; $r_0 = 0$ *indicates solid circular membrane*

		α						
r_0	Method	0	0·5	1·0	1·5	2·0	2·5	3·0
0	DQ	2·4048	2·1827	2·0108	1·8731	1·7598	1·6646	1·5832
	FE	2·4049	2·1828	2·0109	1·8732	1·7600	1·6647	1·5833
	RQs1	2·4048	2·1828	2·0108	1·8732	1·7599	1·6647	1·5832
	RQs2	2·4049	2·1828	2·0108	1·8731	1·7598	1·6646	1·5832
	RQs1/opt	2·4049	2·1828	2·0109	1·8732	1·7599	1·6647	1·5832
	RQs2/opt	2·4049	2·1828	2·0108	1·8731	1·7599	1·6647	1·5832
	LBs2	2·4006	2·1795	2·0084	1·8712	1·7583	1·6634	1·5822
	exact	2·4048	—	—	—	—	—	—
0·1	DQ	3·3204	2·9426	2·6682	2·4578	2·2901	2·1524	2·0368
	FE	3·3182	2·9410	2·6670	2·4569	2·2893	2·1518	2·0362
	RQa	3·3145	2·9379	2·6642	2·4544	2·2870	2·1496	2·0342
	RQa/opt	3·3140	2·9375	2·6640	2·4542	2·2868	2·1495	2·0341
	LBa	—	2·9364	2·6623	2·4523	2·2850	2·1476	2·0322
	exact	3·3139	—	—	—	—	—	—
0·2	DQ	3·8166	3·3476	3·0161	2·7662	2·5694	2·4092	2·2757
	FE	3·8193	3·3498	3·0181	2·7681	2·5711	2·4109	2·2772
	RQa	3·8163	3·3473	3·0158	2·7660	2·5692	2·4091	2·2755
	RQa/opt	3·8160	3·3471	3·0156	2·7659	2·5691	2·4090	2·2754
	LBa	3·8157	3·3458	3·0140	2·7642	2·5675	2·4074	2·2738
	exact	3·8160	—	—	—	—	—	—
0·3	DQ	4·4124	3·8327	3·4330	3·1365	2·9053	2·7186	2·5638
	FE	4·4157	3·8355	3·4355	3·1387	2·9074	2·7205	2·5655
	RQa	4·4127	3·8329	3·4332	3·1366	2·9054	2·7187	2·5638
	RQa/opt	4·4125	3·8328	3·4331	3·1365	2·9054	2·7187	2·5638
	LBa	4·4118	3·8314	3·4316	3·1350	2·9038	2·7172	2·5624
	exact	4·4124	—	—	—	—	—	—
0·4	DQ	5·1830	4·4601	3·9732	3·6169	3·3419	3·1213	2·9391
	FE	5·1867	4·4632	3·9759	3·6194	3·3442	3·1234	2·9412
	RQa	5·1833	4·4602	3·9733	3·6170	3·3420	3·1213	2·9393
	RQa/opt	5·1832	4·4602	3·9733	3·6171	3·3420	3·1214	2·9393
	LBa	5·1822	4·4588	3·9718	3·6156	3·3406	3·1200	2·9380
	exact	5·1831	—	—	—	—	—	—
0·5	DQ	6·2460	5·3259	4·7198	4·2822	3·9472	3·6802	3·4610
	FE	6·2503	5·3295	4·7229	4·2851	3·9499	3·6828	3·4633
	RQa	6·2462	5·3260	4·7198	4·2822	3·9473	3·6803	3·4611
	RQa/opt	6·2463	5·3261	4·7200	4·2824	3·9475	3·6805	3·4612
	LBa	6·2449	5·3245	4·7184	4·2808	3·9460	3·6791	3·4599
	exact	6·2461	—	—	—	—	—	—
0·6	DQ	7·8284	6·6155	5·8336	5·2759	4·8526	4·5172	4·2429
	FE	7·8337	6·6200	5·8374	5·2794	4·8559	4·5202	4·2457
	RQa	7·8285	6·6156	5·8336	5·2760	4·8527	4·5172	4·2429
	RQa/opt	7·8289	6·6160	5·8339	5·2762	4·8529	4·5174	4·2431
	LBa	7·8270	6·6140	5·8320	5·2745	4·8513	4·5159	4·2417
	exact	7·8284	—	—	—	—	—	—

continued

TABLE 1—*continued*.

		α						
r_0	Method	0	0·5	1·0	1·5	2·0	2·5	3·0
0·7	DQ	10·4553	8·7580	7·6857	6·9304	6·3611	5·9125	5·5472
	FE	10·4622	8·7637	7·6909	6·9349	6·3654	5·9165	5·5508
	RQa	10·4553	8·7580	7·6858	6·9304	6·3612	5·9126	5·5472
	RQa/opt	10·4559	8·7585	7·6863	6·9308	6·3616	5·9129	5·5475
	LBa	10·4532	8·7560	7·6840	6·9287	6·3597	5·9112	5·5459
	exact	10·4552	—	—	—	—	—	—
0·8	DQ	15·6981	13·0362	11·3879	10·2394	9·9304	8·7066	8·1599
	FE	15·7085	13·0448	11·3954	10·2462	9·3866	8·7124	8·1653
	RQa	15·7560	13·0878	11·4015	10·2471	9·3848	8·7091	8·1623
	RQa/opt	15·6991	13·0371	11·3887	10·2401	9·3810	8·7072	8·1604
	LBa	15·6950	13·0335	11·3855	10·2372	9·3784	8·7048	8·1582
	exact	15·6981	—	—	—	—	—	—

As an alternative approach, one may start with slightly more complicated admissible functions which allow for optimization of Rayleigh's quotient, viz.,

$$\phi_0 = (1 - r)(1 + \gamma r) \tag{12}$$

or

$$\phi_0 = (1 - r^2)(1 + \gamma r) \tag{13}$$

for the solid circular membrane and

$$\phi_0 = (r - r_0)(1 - r)(1 + \gamma r) \tag{14}$$

for the annular one. Here γ is a parameter that can be used to optimize the value obtained for the Rayleigh quotient, as described in reference [10].

As pointed out in reference [9], a byproduct of the iteration process above is a set of lower bounds. Consider a point $\tilde{r}$ along r at which the displacement does not vanish. For the solid circular membrane, it can be pointed $\tilde{r} = 0$; for the annular case, $\tilde{r} = (1 + r_0)/2$ was used. The lower bound is the value of Ω needed to make $\phi_{i+1}(\tilde{r}) = \phi_i(\tilde{r})$. The reported lower bounds are obtained after three iterations, based on the value of Ω which drives the ratio $\phi_3(\tilde{r})/\phi_2(\tilde{r})$ to unity. Only by starting this process with $\phi_0 = 1 - r^2$ can one obtain a lower bound for the solid circular membrane.

4. APPLICATIONS OF THE DQ METHOD

The application of the DQ method requires the partition of the interval [0, 1] (and $[r_0, 1]$ for the annular case) into subintervals of equal length (see Figures 2(a) and (b)). In the

TABLE 2

Fundamental frequency coefficient Ω_1, case $\rho = \rho_0(1 + \alpha r^2)$; $r_0 = 0$ indicates solid circular membrane

r_0	Method	α = 0	0·5	1·0	1·5	2·0	2·5	3·0
0	DQ	2·4048	2·2819	2·1735	2·0778	1·9925	1·9162	1·8475
	FE	2·4049	2·2820	2·1738	2·0779	1·9927	1·9164	1·8477
	RQs1	2·4048	2·2819	2·1736	2·0778	1·9926	1·9163	1·8475
	RQs2	2·4049	2·2819	2·1736	2·0778	1·9925	1·9162	1·8475
	RQs1/opt	2·4049	2·2819	2·1736	2·0778	1·9925	1·9162	1·8475
	RQs2/opt	2·4049	2·2819	2·1736	2·0778	1·9925	1·9162	1·8475
	LBs2	2·4006	2·2784	2·1708	2·0757	1·9910	1·9152	1·8468
	exact	2·4048	—	—	—	—	—	—
0·1	DQ	3·3204	3·0787	2·8797	2·7130	2·5710	2·4485	2·3416
	FE	3·3182	3·0773	2·8787	2·7122	2·5705	2·4482	2·3414
	RQa	3·3145	3·0740	2·8757	2·7095	2·5679	2·4458	2·3391
	RQa/opt	3·3140	3·0736	2·8754	2·7093	2·5679	2·4458	2·3391
	LBa	—	3·0721	2·8732	2·7067	2·5652	2·4430	2·3364
	exact	3·3139	—	—	—	—	—	—
0·2	DQ	3·8166	3·4974	3·2435	3·0361	2·8630	2·7159	2·5891
	FE	3·8193	3·4998	3·2457	3·0381	2·8649	2·7177	2·5907
	RQa	3·8163	3·4972	3·2433	3·0359	2·8628	2·7157	2·5889
	RQa/opt	3·8160	3·4970	3·2432	3·0359	2·8629	2·7158	2·5890
	LBa	3·8157	3·4953	3·2410	3·0336	2·8606	2·7136	2·5868
	exact	3·8160	—	—	—	—	—	—
0·3	DQ	4·4124	3·9943	3·6730	3·4171	3·2073	3·0314	2·8814
	FE	4·4157	3·9972	3·6757	3·4195	3·2095	3·0336	2·8834
	RQa	4·4127	3·9945	3·6732	3·4172	3·2074	3·0315	2·8815
	RQa/opt	4·4125	3·9944	3·6732	3·4172	3·2075	3·0317	2·8817
	LBa	4·4118	3·9927	3·6711	3·4152	3·2055	3·0298	2·8798
	exact	4·4124	—	—	—	—	—	—
0·4	DQ	5·1830	4·6320	4·2233	3·9053	3·6492	3·4373	3·2584
	FE	5·1867	4·6353	4·2262	3·9081	3·6518	3·4397	3·2606
	RQa	5·1833	4·6322	4·2234	3·9055	3·6493	3·4374	3·2584
	RQa/opt	5·1832	4·6322	4·2235	3·9056	3·6495	3·4377	3·2587
	LBa	5·1822	4·6305	4·2216	3·9037	3·6477	3·4359	3·2570
	exact	5·1831	—	—	—	—	—	—
0·5	DQ	6·2460	5·5070	4·9781	4·5762	4·2576	3·9974	3·7794
	FE	6·2503	5·5108	4·9815	4·5793	4·2606	4·0000	3·7820
	RQa	6·2462	5·5072	4·9782	4·5763	4·2577	3·9974	3·7795
	RQa/opt	6·2463	5·5074	4·9784	4·5765	4·2580	3·9977	3·7798
	LBa	6·2449	5·5054	4·9765	4·5747	4·2563	3·9961	3·7782
	exact	6·2461	—	—	—	—	—	—
0·6	DQ	7·8284	6·8050	6·0984	5·5737	5·1641	4·8333	4·5587
	FE	7·8337	6·8095	6·1025	5·5773	5·1676	4·8365	4·5617
	RQa	7·8285	6·8051	6·0985	5·5737	5·1642	4·8333	4·5587
	RQa/opt	7·8289	6·8054	6·0989	5·5740	5·1646	4·8337	4·5591
	LBa	7·8270	6·8033	6·0968	5·5721	5·1628	4·8320	4·5575
	exact	7·8284	—	—	—	—	—	—

continued

TABLE 2—*continued*.

r_0	Method	α						
		0	0·5	1·0	1·5	2·0	2·5	3·0
0·7	DQ	10·4553	8·9547	7·9557	7·2300	6·6722	6·2263	5·8591
	FE	10·4622	8·9606	7·9610	7·2348	6·6766	6·2304	5·8630
	RQa	10·4553	8·9547	7·9558	7·2301	6·6722	6·2263	5·8591
	RQa/opt	10·4559	8·9553	7·9563	7·2306	6·6727	6·2267	5·8596
	LBa	10·4532	8·9526	7·9539	7·2283	6·6707	6·2248	5·8578
	exact	10·4552	—	—	—	—	—	—
0·8	DQ	15·6981	13·2394	11·6616	10·5397	9·6895	9·0165	8·4669
	FE	15·7085	13·2482	11·6693	10·5467	9·6959	9·0225	8·4724
	RQa	15·7560	13·2607	11·6735	10·5403	9·6908	9·0158	8·4673
	RQa/opt	15·6991	13·2403	11·6624	10·5405	9·6902	9·0173	8·4674
	LBa	15·6950	13·2366	11·6591	10·5375	9·6875	9·0147	8·4651
	exact	15·6981	—	—	—	—	—	—

case of the solid circular membrane one adds the condition $W'(0) = 0$. Following Bert's well estabilshed notation [6] and making use of the DQ method one obtains:

(1) the solid circular membrane;

$$\sum_{k=1}^{N-1} A_{1k} W_k = 0, \qquad \sum_{k=1}^{N-1} (r_i B_{ik} + A_{ik}) W_k + \Omega^2 f(r_i) r_i W_i = 0, \qquad (i = 2, \ldots, N-1); \tag{15}$$

(2) the annular membrane,

$$\sum_{k=2}^{N-1} (r_i B_{ik} + A_{ik}) W_k + \Omega^2 f(r_i) r_i W_i = 0, \qquad (i = 2, \ldots, N-1). \tag{16}$$

The coefficients A_{ik} and B_{ik} are linear combinations corresponding to the first and second derivatives, respectively [6]. The fundamental frequency coefficient, Ω_1, was determined making $N = 9$ while the second natural frequency, Ω_2, was evaluated taking $N = 14$. To our knowledge, the DQ method has not been proven to provide either an upper or a lower bound.

5. NUMERICAL RESULTS

Tables 1, 2 and 3 depict values of Ω_1 for $n = 1, 2$ and 3, respectively, for solid circular and annular membranes as a function of r_0 and α. Results obtained from the finite element method are indicated by FE; those obtained from the differential quadrature method are indicated with DQ. For application of the FE method [7], only results for the fundamental frequency coefficient Ω_1 are presented. These results are based on taking 25 elements. For the solid circular membranes, those results obtained by starting from $\phi_0 = 1 - r$ are denoted by RQs1, while those starting from $\phi_0 = 1 - r^2$ are denoted by RQs2. For the annular membrane, the Rayleigh quotient results are denoted by RQa. In all cases except $n = 3$, the Rayleigh quotient result is based on use of ϕ_2; for $n = 3$, the Rayleigh quotient

TABLE 3

*Fundamental frequency coefficient Ω_1, case $\rho = \rho_0(1 + \alpha r^3)$; $r_0 = 0$ indicates solid circular membrane, and * indicates that results are based on ϕ_1 (not ϕ_2)*

		α						
r_0	Method	0	0·5	1·0	1·5	2·0	2·5	3·0
0	DQ	2·4048	2·3311	2·2620	2·1971	2·1363	2·0794	2·0259
	FE	2·4049	2·3313	2·2621	2·1972	2·1364	2·0795	2·0261
	RQs1	2·4048	2·3312	2·2620	2·1971	2·1363	2·0794	2·0260
	RQs2	2·4049	2·3312	2·2620	2·1971	2·1363	2·0793	2·0258
	Rqs1/opt	2·4049	2·3312	2·2620	2·1971	2·1363	2·0793	2·0259
	Rqs2/opt	2·4049	2·3312	2·2620	2·1971	2·1363	2·0793	2·0259
	LBs2	2·4006	2·3274	2·2587	2·1944	2·1342	2·0778	2·0249
	exact	2·4048	—	—	—	—	—	—
0·1	DQ	3·3204	3·1603	3·0178	2·8903	2·7762	2·6733	2·5803
	FE	3·3182	3·1588	3·0166	2·8895	2·7755	2·6728	2·5798
	RQa*	3·3241	3·1627	3·0189	2·8907	2·7759	2·6726	2·5791
	Rqa/opt	3·3140	3·1550	3·0131	2·8864	2·7727	2·6702	2·5775
	LBa	—	3·1536	3·0108	2·8835	2·7695	2·6670	2·5743
	exact	3·3139	—	—	—	—	—	—
0·2	DQ	3·8166	3·5948	3·4034	3·2370	3·0912	2·9625	2·8477
	FE	3·8193	3·5972	3·4056	3·2391	3·0932	2·9642	2·8494
	RQa*	3·8228	3·5990	3·4062	3·2389	3·0925	2·9631	2·8481
	RQa/opt	3·8160	3·5942	3·4030	3·2368	3·0911	2·9623	2·8477
	LBa	3·8157	3·5926	3·4007	3·2342	3·0884	2·9597	2·8452
	exact	3·8160	—	—	—	—	—	—
0·3	DQ	4·4124	4·1072	3·8533	3·6389	3·4550	3·2956	3·1557
	FE	4·4157	4·1102	3·8560	3·6413	3·4573	3·2977	3·1577
	RQa*	4·4173	4·1103	3·8553	3·6400	3·4558	3·2960	3·1559
	RQa/opt	4·4125	4·1073	3·8534	3·6390	3·4553	3·2959	3·1560
	LBa	4·4118	4·1055	3·8512	3·6367	3·4530	3·2937	3·1540
	exact	4·4124	—	—	—	—	—	—
0·4	DQ	5·1830	4·7606	4·4227	4·1456	3·9134	3·7155	3·5445
	FE	5·1867	4·7639	4·4257	4·1483	3·9160	3·7181	3·5469
	RQa*	5·1867	4·7627	4·4240	4·1463	3·9139	3·7159	3·5447
	RQa/opt	5·1832	4·7608	4·4230	4·1459	3·9138	3·7161	3·5450
	LBa	5·1822	4·7589	4·4208	4·1438	3·9118	3·7142	3·5433
	exact	5·1831	—	—	—	—	—	—
0·5	DQ	6·2460	5·6512	5·1953	4·8325	4·5353	4·2864	4·0742
	FE	6·2503	5·6550	5·1987	4·8357	4·5348	4·2893	4·0769
	RQa*	6·2488	5·6527	5·1961	4·8330	4·5357	4·2868	4·0744
	RQa/opt	6·2463	5·6515	5·1956	4·8329	4·5358	4·2870	4·0748
	LBa	6·2449	5·6495	5·1935	4·8309	4·5340	4·2853	4·0732
	exact	6·2461	—	—	—	—	—	—
0·6	DQ	7·8284	6·9644	6·3316	5·8433	5·4523	5·1301	4·8588
	FE	7·8337	6·9691	6·3358	5·8473	5·4560	5·1336	4·8621
	RQa*	7·8306	6·9656	6·3323	5·8439	5·4527	5·1305	4·8591
	RQa/opt	7·8289	6·9649	6·3321	5·8440	5·4529	5·1308	4·8595
	LBa	7·8270	6·9626	6·3299	5·8419	5·4511	5·1291	4·8579
	exact	7·8284	—	—	—	—	—	—

continued

TABLE 3—*continued.*

r_0	Method	α: 0	0·5	1·0	1·5	2·0	2·5	3·0
0·7	DQ	10·4553	9·1290	8·2032	7·5107	6·9680	6·5278	6·1616
	FE	10·4622	9·1350	8·2086	7·5157	6·9726	6·5322	6·1657
	RQa*	10·4570	9·1299	8·2039	7·5113	6·9685	6·5283	6·1620
	RQa/opt	10·4559	9·1296	8·2038	7·5114	6·9686	6·5285	6·1623
	LBa	10·4532	9·1268	8·2013	7·5091	6·9666	6·5266	6·1605
	exact	10·4552	—	—	—	—	—	—
0·8	DQ	15·6981	13·4278	11·9212	10·8288	9·9901	9·3200	8·7689
	FE	15·7085	13·4367	11·9291	10·8359	9·9966	9·3262	8·7747
	RQa*	15·6998	13·4289	11·9221	10·8295	9·9907	9·3207	8·7695
	RQa/opt	15·6991	13·4287	11·9221	10·8296	9·9909	9·3209	8·7697
	LBa	15·6950	13·4249	11·9187	10·8265	9·9881	9·3183	8·7673
	exact	15·6981	—	—	—	—	—	—

result is based on use of ϕ_1, due to memory limitations of the computer. For the optimized Rayleigh quotient, for the solid circular membrane, results obtained by starting from $\phi_0 = (1 - r)(1 + \gamma r)$ are denoted by RQs1/opt, while those obtained by starting from $\phi_0 = (1 - r^2)(1 + \gamma r)$ are denoted by RQs2/opt. Optimized Rayleigh quotient results obtianed for the annular membrane are denoted by RQa/opt. For the annular membrane, the optimized Rayleigh quotient results are based on the use of ϕ_1 with γ chosen to minimize $\mathscr{R}$. The lower bounds for the solid circular case are denoted by LBs2 for the starting function $\phi_0 = 1 - r^2$ and by LBa for the annular case.

TABLE 4

Second frequency coefficient Ω_2, *case* $\rho = \rho_0\ (1 + \alpha r)$; $r_0 = 0$ *indicates solid circular membrane*

r_0	Method	α: 0	0·5	1·0	1·5	2·0	2·5	3·0
0	DQ	5·5201	4·9630	4·5549	4·2374	3·9802	3·7659	3·5834
	exact	5·5201	—	—	—	—	—	—
0·1	DQ	6·8588	6·0786	5·5203	5·0939	4·7541	4·4747	4·2397
	exact	6·8576	—	—	—	—	—	—
0·2	DQ	7·7856	6·8320	6·1640	5·6616	5·2654	4·9426	4·6728
	exact	7·7855	—	—	—	—	—	—
0·3	DQ	8·9328	7·7635	6·9616	6·3671	5·9033	5·5281	5·2166
	exact	8·9328	—	—	—	—	—	—
0·4	DQ	10·4433	8·9908	8·0157	7·3026	6·7514	6·3090	5·9435
	exact	10·4432	—	—	—	—	—	—
0·5	DQ	12·5468	10·7022	9·4893	8·6138	7·9433	7·4085	6·9691
	exact	12·5469	—	—	—	—	—	—
0·6	DQ	15·6948	13·2662	11·7018	10·5865	9·7393	9·0678	8·5185
	exact	15·6948	—	—	—	—	—	—
0·7	DQ	20·9354	17·5390	15·3946	13·8835	12·7449	11·8472	11·1159
	exact	20·9355	—	—	—	—	—	—
0·8	DQ	31·4157	26·0900	22·7928	20·4952	18·7766	17·4285	16·3347
	exact	31·4110	—	—	—	—	—	—

TABLE 5

Second frequency coefficient Ω_2, case $\rho = \rho_0\ (1 + \alpha r^2)$; $r_0 = 0$ indicates solid circular membrane

		α						
r_0	Method	0	0·5	1·0	1·5	2·0	2·5	3·0
0	DQ	5·5201	5·1412	4·8416	4·5969	4·3914	4·2149	4·0609
	exact	5·5201	—	—	—	—	—	—
0·1	DQ	6·8588	6·3205	5·9020	5·5629	5·2798	5·0379	4·8277
	exact	6·8576	—	—	—	—	—	—
0·2	DQ	7·7856	7·1061	6·5876	6·1730	5·8307	5·5412	5·2917
	exact	7·7855	—	—	—	—	—	—
0·3	DQ	8·9328	8·0660	7·4189	6·9101	6·4955	6·1486	5·8526
	exact	8·9328	—	—	—	—	—	—
0·4	DQ	10·4433	9·3185	8·5003	7·8687	7·3613	6·9417	6·5870
	exact	10·4432	—	—	—	—	—	—
0·5	DQ	12·5468	11·0526	9·9958	9·1962	8·5634	8·0460	7·6126
	exact	12·5469	—	—	—	—	—	—
0·6	DQ	15·6948	13·6365	12·2255	11·1796	10·3640	9·7045	9·1570
	exact	15·6948	—	—	—	—	—	—
0·7	DQ	20·9354	17·9268	15·9312	14·4826	13·3692	12·4787	11·7455
	exact	20·9355	—	—	—	—	—	—
0·8	DQ	31·4157	26·4927	23·3378	21·0952	19·3957	18·0504	16·9510
	exact	31·4110	—	—	—	—	—	—

TABLE 6

Second frequency coefficient Ω_2, case $\rho = \rho_0\ (1 + \alpha r^3)$; $r_0 = 0$ indicates solid circular membrane

		α						
r_0	Method	0	0·5	1·0	1·5	2·0	2·5	3·0
0	DQ	5·5201	5·2374	4·9993	4·7969	4·6227	4·4709	4·3370
	exact	5·5201	—	—	—	—	—	—
0·1	DQ	6·8588	6·4548	6·1241	5·8474	5·6108	5·4048	5·2230
	exact	6·8576	—	—	—	—	—	—
0·2	DQ	7·7856	7·2698	6·8554	6·5123	6·2211	5·9692	5·7477
	exact	7·7855	—	—	—	—	—	—
0·3	DQ	8·9328	8·2618	7·7336	7·3023	6·9402	6·6299	6·3594
	exact	8·9328	—	—	—	—	—	—
0·4	DQ	10·4433	9·5486	8·8615	8·3102	7·8539	7·4673	7·1340
	exact	10·4432	—	—	—	—	—	—
0·5	DQ	12·5468	11·3181	10·4014	9·6817	9·0958	8·6063	8·1889
	exact	12·5469	—	—	—	—	—	—
0·6	DQ	15·6948	13·9377	12·6719	11·7021	10·9273	10·2895	9·7523
	exact	15·6948	—	—	—	—	—	—
0·7	DQ	20·9354	18·2626	16·4135	15·0345	13·9547	13·0789	12·3502
	exact	20·9355	—	—	—	—	—	—
0·8	DQ	31·4157	26·8610	23·8491	21·6673	19·9929	18·6551	17·5546
	exact	31·4110	—	—	—	—	—	—

The agreement between the results obtianed by means of the DQ technique and the FE method is very good. For $r_0 = 0{\cdot}1$ the FE element yields results which are very slightly lower than those obtained by means of the DQ method. This situation reverses for values of $r_0 \geqslant 0{\cdot}2$. The fundamental eigenvalues obtained using the Rayleigh quotient are in excellent agreement with the DQ and FE results. Results indicate that sometimes the DQ results are just above the exact solution and sometimes just below it. The Rayleigh quotient results are sometimes better (i.e., lower) than those from FE for the fundamental frequency. The various RQ/opt results are seen to many times yield essentially the same or somewhat more accurate (i.e., lower) results than were obtained from RQ; the RQ/opt results require substantially less symbolic computing effort. Finally, the lower bound method failed to yield a lower bound for $n = 1$, $\alpha = 0$, and $r_0 = 0{\cdot}1$. The reason for this is unknown, but it may be due to an erroneous result from the symbolic manipulator.

Tables 4, 5 and 6 present results for the second eigenvalue obtained by means of the DQ method. It is important to point out that for $\alpha = 0$, the DQ method results are in excellent agreement with the exact second eigenvalues, available only for $\alpha = 0$ and shown in the tables. The relative error is of the order of 1% and takes place for $r_0 = 0{\cdot}8$ and $\alpha = 3$, as shown in three tables.

6. CONCLUDING REMARKS

The present paper has dealt with solid circular and annular membranes where the density varies with the radial variable linearly, quadratically, and cubically. The approximate techniques employed are quite adequate and efficient for determining the lower axisymmetric frequencies. The differential quadrature and improved/optimized Rayleigh quotient methods are the most accurate overall. The latter provides upper bounds for only the fundamental frequency, while the former yields accurate estimates for multiple frequencies. The optimized Rayleigh quotient method based on iteration of improvement yields quite comparable results to the improved Rayleigh quotient based on two iterations of improvement, but with a significant reduction in symbolic computing effort.

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